Viète's Formula in "A Fractal Eigenvector"

RHETT S. ROBINSON, Independent researcher, USA

Abstract. This paper explores a relationship of the asymptotic behavior of the leading element of eigenvectors belonging to the dominant eigenvalue of a recursively-constructed family of Mandelbrot matrices to Viète's formula, helping to explain the appearance of π in this element.

CCS Concepts: • **Mathematics of computing** → **Mathematical analysis**.

Additional Key Words and Phrases: Mandelbrot matrices, asymptotic behavior

Recommended Reference Format:

Rhett S. Robinson. 2021. Viète's Formula in "A Fractal Eigenvector". *Maple Trans.* 1, 2, Article 14367 (November 2021), 3 pages. <https://doi.org/10.5206/mt.v1i2.14367>

1 Introduction and Background

In [1], a recursively-defined family of Mandelbrot matrices are constructed with characteristic polynomial $C_n(\lambda)$ satisfying $C_{n+1} = \lambda C_n^2 - 1$ with $C_0 = 1$ (see Remark under Proposition 2), and using ρ_n to refer to the dominant eigenvalue of C_n . We will follow this convention, so that $\rho_n > 0$.

1.1 The asymptotic equation

The eigenvector associated with this eigenvalue is explored, using the eigenvector normalization as defined in equation (7) of [1]. In equation (15) of [1] the leading entry is defined, along with an asymptotic equality that was established only by high-precision numerical experiments, reproduced here:

$$x_{n+1,1}(\rho_{n+1}) = \rho_{n+1}^n \prod_{k=1}^n C_k(\rho_{n+1}) = \frac{2^{n+1}}{\pi} \left(1 + \tilde{O}(4^{-n})\right)$$

where $\tilde{O}(g(n))$ is shorthand for $O(g(n) \log^m g(n))$ for some fixed m .

1.2 Viète's formula

Viète's formula is an infinite product that converges to $2/\pi$, defined by

$$\begin{aligned} v_1 &= 1 \\ v_k &= \sqrt{2 + v_{k-1}} \\ \lim_{n \rightarrow \infty} \prod_{k=1}^n \frac{v_k}{2} &= \frac{2}{\pi}. \end{aligned}$$

Author's address: Rhett S. Robinson, rrhett@gmail.com, Independent researcher, Kirkland, USA.

Permission to make digital or hard copies of all or part of this work for any use is granted without fee, provided that copies bear this notice and the full citation on the first page. Copyrights for third-party components of this work must be honored. © 2021 Copyright held by the owner/author(s). Publication rights licensed to Maple Transactions, under Creative Commons CC-BY 4.0 License.

<https://doi.org/10.5206/mt.v1i2.14367>

2 Finding Viète's formula

For convenience, we will denote $C_{k,n} = C_k(\rho_{n+1})$. First we make a few observations on $C_{k,n}$ and ρ_n .

LEMMA 2.1. $\rho_n < \rho_{n+1} < 2$.

PROOF. First we will show that $\rho_n < \rho_{n+1}$. We have $C_1 = \lambda - 1$ so that $\rho_1 = 1$. By the recurrence, we have $C_2 = \lambda(\lambda - 1)^2 - 1$ so that $C_2(1) = -1$ and $C_2(2) = 1$. This implies there is a root of C_2 greater than 1. ρ_2 is the largest root of C_2 so it follows that $\rho_2 > 1 = \rho_1$. Assume for induction that $\rho_k > \rho_{k-1}$. By the recurrence, $C_{k+1}(\rho_k) = \rho_k C_k^2(\rho_k) - 1 = -1$. Because C_k is a polynomial with positive leading coefficient, we know there is a root greater than ρ_k , and as ρ_{k+1} is the greatest root of C_{k+1} , we have that $\rho_{k+1} > \rho_k$.

Second, we will show that $\rho_k < 2$. We will show this by proving that $C_k(\lambda) \geq 1$ for $\lambda \geq 2$. Because $C_k(\rho_k) = 0$ by definition, from this it follows that $\rho_k < 2$. Note that $C_1 = \lambda - 1 \geq 1$ for $\lambda \geq 2$. Assume for induction that $C_k \geq 1$ for $\lambda \geq 2$. This implies $C_{k+1} = \lambda C_k^2 - 1 \geq 2 \cdot 1^2 - 1 = 1$ for $\lambda \geq 2$ establishing our result. $\square$

We will also assume $\lim_{n \rightarrow \infty} \rho_n = 2$ based on the empirical asymptotic expansion given by equation (6) in [1].

LEMMA 2.2. $C_{k,n} > 0$ for $1 \leq k \leq n$.

PROOF. $C_{k,n} = C_k(\rho_{n+1})$. For $k \leq n$, $\rho_{n+1} > \rho_k$, and C_k being a polynomial with positive leading coefficient and largest zero at ρ_k implies $C_k(\rho_{n+1}) > 0$. $\square$

Define sequence $\{a_{k,n}\}$ by $a_{k,n} = \rho_{n+1} C_{k,n}$, so that

$$x_{n+1,1}(\rho_{n+1}) = \prod_{k=1}^n a_{k,n}.$$

Note that the recurrence relationship gives

$$C_{k-1,n} = \frac{1}{\sqrt{\rho_{n+1}}} \sqrt{1 + C_{k,n}}.$$

Thus

$$\begin{aligned} a_{k-1,n} &= \rho_{n+1} C_{k-1,n} \\ &= \sqrt{\rho_{n+1}} \sqrt{1 + C_{k,n}} \\ &= \sqrt{\rho_{n+1} + \rho_{n+1} C_{k,n}} \\ &= \sqrt{\rho_{n+1} + a_{k,n}} \end{aligned}$$

Because $\lim_{n \rightarrow \infty} \rho_{n+1} = 2$ we have the recurrence

$$\lim_{n \rightarrow \infty} a_{k-1,n} = \sqrt{2 + \lim_{n \rightarrow \infty} a_{k,n}}.$$

Next, consider the sequence $\{b_{k,n}\}$ defined by $b_{k,n} = a_{n-k+1,n}$ and the limit of its product

$$\lim_{n \rightarrow \infty} \prod_{k=1}^n \frac{b_{k,n}}{2}.$$

Written this way, we have $b_{1,n} = a_{n,n} = \sqrt{\rho_{n+1}}$. Since $\rho_{n+1} < 2$, we have $b_{1,n} < v_1$. Assume $b_{k-1,n} < v_{k-1}$. For a general term

$$b_{k,n} = a_{n-k+1,n} = \sqrt{\rho_{n+1} + a_{n-k+2,n}} = \sqrt{\rho_{n+1} + b_{k-1,n}}.$$

Because $\rho_{n+1} < 2$, and $b_{k-1,n} < v_{k-1}$ inductively, we have

$$b_{k,n} < \sqrt{2 + b_{k-1,n}} < \sqrt{2 + v_{k-1}} = v_k.$$

Thus we can establish $b_{k,n} < v_k$ for all k, n . It is also easy to see that since ρ_{n+1} increases with n , $b_{k,n}$ also increases with n .

Note that $\lim_{n \rightarrow \infty} b_{1,n} = \sqrt{2} = v_1$. Assume $\lim_{n \rightarrow \infty} b_{k-1,n} = v_{k-1}$. Then we see

$$\begin{aligned} \lim_{n \rightarrow \infty} b_{k,n} &= \lim_{n \rightarrow \infty} \sqrt{\rho_{n+1} + b_{k-1,n}} \\ &= \sqrt{\lim_{n \rightarrow \infty} \rho_{n+1} + \lim_{n \rightarrow \infty} b_{k-1,n}} \\ &= \sqrt{2 + v_{k-1}}. \end{aligned}$$

Thus $\lim_{n \rightarrow \infty} b_{k,n} = v_k$, and so this reordering of the terms in the product reduces to Viète's formula in the limit.

Let $\{B_n\}$ be the partial products of $b_{k,n}/2$, and $\{V_n\}$ be the partial products of $v_k/2$. Note that from $b_{k,n} < v_k$ it trivially follows that $B_k < V_k$. It appears that B_k is increasing after B_3 from which it would follow that $\{B_n\}$ converges, but we have not yet established a proof of this.

Now knowing this relationship to Viète's formula, we might explore some other of the findings in [1]. For example, it is stated that the lower entries of $x_k(\rho_{k+1})$ converge to A048896. Interestingly, this sequence is also the numerators of Maclaurin series for $1 - \left(\frac{\sin x}{x}\right)^2$ [2], and Viète's formula can be obtained from the sinc function

$$\frac{\sin x}{x} = \cos \frac{x}{2} \cdot \cos \frac{x}{4} \cdot \cos \frac{x}{8} \cdot \dots$$

evaluated at $x = \pi/2$, and using the half-angle formula

$$\cos \frac{x}{2} = \sqrt{\frac{1 + \cos x}{2}}$$

to expand each term on the right as a function of earlier terms [3].

Additionally, we see there exists a simple relationship between A001316 and A048896: $A048896(n) = (1/2) \cdot A001316(n+1)$. Given the scaling factor relating the two halves of the eigenvector, it is perhaps unsurprising we would find a similar relationship between two sequences already observed in the eigenvectors.

References

- [1] Neil J. Calkin, Eunice Y. S. Chan, Robert M. Corless, David J. Jeffrey, and Piers W. Lawrence. "A Fractal Eigenvector". In: *American Mathematical Monthly* (To appear). URL: <https://arxiv.org/abs/2104.01116> (cit. on pp. 1–3).
- [2] OEIS Foundation Inc. *The On-Line Encyclopedia of Integer Sequences*. [Online; accessed 25-August-2021]. 2021. URL: <http://oeis.org/A048896> (cit. on p. 3).
- [3] Wikipedia contributors. *Viète's formula* — *Wikipedia, The Free Encyclopedia*. [Online; accessed 25-August-2021]. 2021. URL: https://en.wikipedia.org/w/index.php?title=Vi%C3%A8te%27s_formula&oldid=1040174420 (cit. on p. 3).