Problems, Puzzles, and Challenges

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Abstract. We give here some problems and puzzles that need a combination of thought and computation to solve. Readers are invited to submit their solutions to the journal at mapletransactions.org. Please include the problem number with your submission. Readers are also invited to submit their own problems for possible inclusion in future issues.

Additional Key Words and Phrases: Logarithmic capacity, Mandelbrot polynomials, unimodal sequence, Fibonacci-number matrices, inverse symbolic calculator, high-precision fraud

Recommended Reference Format:

David J. Jeffrey. 2021. Problems, Puzzles, and Challenges . *Maple Trans.* 1, 2, Article 14351 (November 2021), 4 pages. <https://doi.org/10.5206.mt.v1i2.14351>

Here are some challenges with a computational flavour. They are not necessarily purely challenges in computation, but more generally we are interested in problems that benefit from the insights computers can give. Some are inspired by publications in this issue, but problems from anywhere are of interest. We welcome submissions from readers.

14351.1. The paper of Baddoo & Trefethen in this journal [1] describes the computation of logarithmic capacity: a basic measure of the size of a set. The abstract states “...it should be possible to compute many more digits if desired in Maple ...” That is our first challenge. The paper presents computations of 16 shapes. Our second challenge is to apply their methods to a more complicated shape, namely, the Maple logo. You are free to choose which version of the logo you study.

The venerable	<pre> \ ^ / </pre>	
ASCII logo from	<pre>._ \ / _.</pre>	
command-line	<pre>\ MAPLE /</pre>	or the modern
Maple	<pre><_ _ _ _ _></pre>
	<pre> </pre>	

14351.2. A paper with the inviting word “conjectures” in the title appeared in [3]. One conjecture we highlight is whether the Mandelbrot polynomials are unimodal. Further, are they (or possibly a subset of them) log-concave?

The Mandelbrot polynomials are defined in [3] to be

$$p_0(z) = 0, \quad p_{n+1}(z) = p_n^2(z) + z, \quad n > 0.$$

An example is

$$p_4(z) = z^8 + 4z^7 + 6z^6 + 6z^5 + 5z^4 + 2z^3 + z^2 + z.$$

Notice that the coefficients at either end are 1, and that the largest coefficients appear in the middle – a graph of the coefficients of $p_7(z)$ is

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<https://doi.org/10.5206.mt.v1i2.14351>

The meaning of “unimodal” is that the graph contains a single maximum. That is, there is an index K such that

$$\begin{aligned} a_n &\leq a_{n+1}, & n &\leq K, \\ a_n &\geq a_{n+1}, & n &\geq K. \end{aligned}$$

The max might occur twice as seen in p_4 . A stronger property is log-concavity. A polynomial is log-concave if its coefficients obey $a_{n-1}a_{n+1}/a_n^2 < 1$.

Both conjectures are open, as far as we know.

14351.3. The Sequence Vanishes (apologies to Hitchcock for the title). Bill Gosper (a constant source of puzzles and observations) reports a sequence of determinants that mysteriously vanishes abruptly [4]. It can be generated by the Maple commands below.

```
with(combinat):
with(LinearAlgebra):
F := n -> Matrix(n,n,(i,j)-> fibonacci(i+1) mod (j+1)):
Vanishes:= n -> Determinant(F(n)):
```

>F(1)

$$\begin{bmatrix} 1 \end{bmatrix} \quad (1)$$

>F(2)

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \quad (2)$$

>F(3)

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 1 & 0 & 3 \end{bmatrix} \quad (3)$$

>F(4)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 1 & 0 & 3 & 3 \\ 1 & 2 & 1 & 0 \end{bmatrix} \quad (4)$$

> seq(*Vanishes*(*k*), *k* = 1..34);

$$\begin{aligned} &1, 2, 6, -6, -84, -468, -2304, 4032, -40320, 187200, 9469440, \\ &23483520, 518918400, 9482054400, 40178073600, \\ &770466816000, -61318301184000, 409687455744000, \\ &-15044447416320000, 118149165582336000, \\ &-536843380727808000, 33991042134196224000, \\ &-7534439454413684736000, -36871569141860597760000, \\ &988282095197340303360000, -31916582708804859985920000, \\ &887427632197884767109120000, \\ &114416633221928350911037440000, \\ &3096872763952273311714508800000, \\ &187571217810512387633106124800000, \\ &-4039374173618494181047114137600000, \\ &-78680486410631223667290066124800000, \\ &-166181436312725157334271970508800000, \\ &59155680772310826436629853372416000000 \end{aligned} \quad (5)$$

> seq(*Vanishes*(*k*), *k* = 35..55)

$$0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \quad (6)$$

From $n = 35$ onwards, it seems to be zero. Is this true?

14351.4. The Maple `identify` command takes as its argument a floating-point number and searches for a symbolic equivalent. Thus `identify(4.14159265)` returns $1 + \pi$. Such identifications may have a venerable history: it has been suggested that Stirling obtained the $\sqrt{2\pi}$ factor in his approximation to factorial by computation [7]. Current alternatives to `identify` are discussed in [6].

Any identification might give a false positive, of course, and examples of potential false positives were given by the Borwein brothers under the eye-catching title “high-precision fraud” [2]. Bill Gosper reports remarkable agreement between the expressions below [5]. If it is a false positive, how does one find such amazing agreement?

$$\Gamma(1/4) = \frac{1}{2} \left(1 + 5^{\frac{1}{4}}\right)^{\frac{3}{2}} \sqrt{5} \sqrt{1 + \sqrt{2}} \sqrt{\sqrt{2} + 5^{\frac{1}{4}}} \left(\sqrt{5} + 1\right)^{\frac{11}{4}} \left(3 + \sqrt{10}\right)^{\frac{1}{4}} e^{-\frac{65\pi}{3}} \pi^{\frac{3}{4}} \sinh(20\pi) 2^{\frac{5}{16}}. \quad (7)$$

In Maple syntax, the equation is

```
GAMMA(1/4) = (1/2)*(1+5^(1/4))^(3/2)*sqrt(5)*sqrt((1+sqrt(2)))*
sqrt(sqrt(2)+5^(1/4))* (1+sqrt(5))^(11/4)* (3+sqrt(10))^(1/4)*
exp(-65*Pi/3)*Pi^(3/4)*sinh(20*Pi)*2^(5/16)
```

References

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